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STABILITY OF HARMONIC MAPS OF KÄHLER MANIFOLDS

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1. Introduction

Harmonic maps [6] φ between Riemannian manifolds (M, g) and (N, h) are the critical points of the Dirichlet energy integral

$$E(\varphi) = \frac{1}{2}\int_M |d\varphi|^2 \, d\,\mathrm{vol}.$$

When M and N are Hermitian manifolds Lichnerowicz [9] observed that the energy E decomposes into two parts E' and E'' corresponding with the parts of the differential $d\varphi$ acting in holomorphic and antiholomorphic tangents. The difference $E' - E''$ can be expressed in terms of the Kähler forms ω^M and ω^N as

$$E'(\varphi) - E''(\varphi) = \int_M \langle \varphi^*\omega^N, \omega^M \rangle \, d\,\mathrm{vol},$$

and this is a homotopy invariant provided that ω^M is coclosed and ω^N is closed. In this case maps for which either E'' or E' vanishes are absolute minima of the energy in their homotopy class and hence are stable harmonic maps. They are of course the holomorphic and antiholomorphic maps between M and N. For simplicity we shall refer to them as $\pm$holomorphic maps. These remarks apply in particular to the case where M and N are both Kähler manifolds.

In general we shall say that a harmonic map φ is (weakly) stable if the second variation of the energy is nonnegative:

$$H_\varphi(v, v) = \frac{d^2}{dt^2} E(\varphi_t)|_{t=0} \geq 0$$

for all smooth variations φ_t of φ, where $v = \dot\varphi_0$. It may be conjectured that there are no other stable harmonic maps between Kähler manifolds besides the $\pm$holomorphic ones. The problem has recently received a lot

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of attention, and when M is a closed Riemann surface and N a Hermitian symmetric space a number of results are known. It is clear that we should assume N is irreducible, otherwise we can find trivial counterexamples by taking a map which is holomorphic into one factor and anti-holomorphic into another. If N has nonpositive curvature, then all harmonic maps are stable, and if M is not simply-connected there are harmonic maps which are not $\pm$holomorphic. Thus from now on we shall concentrate on target manifolds which have metrics of positive curvature. In the following our Hermitian symmetric spaces are all assumed to be compact with simple isometry group; these are the irreducible Hermitian symmetric spaces of compact type, and we shall refer to them as the compact simple Hermitian symmetric spaces.

The simplest case to consider is maps from S^2 to CP^n. An affirmative answer is given by Siu-Yau [19] and in the physics literature [21] where this is studied as the nonlinear σ-model. For Riemann surfaces of nonzero genus Siu [16] extended this result under the assumption of sufficient ramification of φ. However Burns and de Bartolomeis [2] have shown that no such assumption is necessary. (See also Lawson-Simons [8] for an analogous result in the context of minimal currents in CP^n.)

For targets other than CP^n Siu [17] has used a curvature condition to show that all stable harmonic maps from S^2 into the classical irreducible Hermitian symmetric spaces are $\pm$holomorphic, and Zhong [22] has checked that the curvature condition holds also for the exceptional spaces. This case-by-case checking can be avoided by using the formula of Moore [10] for H_φ. This has been done by Burstall, Rawnsley and Salamon [3] to give a uniform proof that the stable harmonic maps from S^2 into any irreducible Hermitian symmetric space are $\pm$holomorphic. This last method also describes all the stable harmonic 2-spheres in all irreducible symmetric spaces.

In fact no assumption on the genus of the Riemann surface is necessary and the range can be any compact simple hermitian symmetric space. We shall give a proof of the following:

Theorem 1. *Let $\varphi: M \to N$ be a stable harmonic map of a closed Riemann surface into a compact simple Hermitian symmetric space. Then φ is $\pm$holomorphic.*

Our method is based on that of Lawson and Simons, which derives a curvature condition by averaging over certain gradient vector fields. The assumption of stability then leads to a simple algebraic condition on the differential of φ. Differentiating this condition allows us to conclude that an open set in the domain carries a Hermitian structure with coclosed

Kähler form. If the domain has dimension four, this form is then necessarily closed and globally defined giving:

Theorem 3. *Let $\varphi: M \to N$ be a stable harmonic map from a real analytic four-dimensional Riemannian manifold M to a compact simple Hermitian symmetric space N, and suppose that there is a point where $d\varphi$ has rank at least three. Then M has a unique Kähler structure $\tilde{J}$ with respect to which φ is holomorphic.*

Ohnita and Udagawa [12] have also obtained Theorem 1 using the Lawson-Simons method, but referring to a study of the curvature of Hermitian symmetric spaces due to Borel [1] and Calabi-Vesentini [5] to deduce the result. However our analysis of the algebraic condition provides more information, and allows us to obtain results for stability of maps from projective spaces generalizing Ohnita [11] for projective spaces as target.

Theorem 5. *Let $\varphi: CP^n \to N$ be a stable harmonic map into a compact simple Hermitian symmetric space. Then φ is $\pm$holomorphic.*

2. Variational formulas

The first part of the proof is valid for a general class of homogeneous Kähler manifolds as target and any domain, so in this more general context we shall begin by discussing the second variation of the energy:

$$H_\varphi(v, v) = (J_\varphi v, v),$$

where $(u, v) = \int_M \langle u, v\rangle \, d\,\mathrm{vol}$, and the Jacobi operator J_φ is given by

$$J_\varphi v = \nabla^* \nabla v - \mathrm{Ric}^\varphi(v)$$

with $\mathrm{Ric}^\varphi(v) = \sum_i R^N(v, d\varphi(e_i))\, d\varphi(e_i)$ for a local orthonormal frame $e_1, \cdots, e_m$ on M. Ric^φ is an endomorphism of the pull-back of the tangent bundle of N and is a generalization of the Ricci tensor.

Let $\mathfrak{g}$ denote the Lie algebra of Killing vector fields on N. Then since isometries preserve the energy, the fields $X \in \mathfrak{g}$ will satisfy $J_\varphi X = 0$.

Now suppose N is Kähler, and consider the vector fields JX (gradient fields when N is Hermitian symmetric). We define a quadratic form Q on $\mathfrak{g}$ by

$$Q(X, X) = H_\varphi(JX, JX) = (J_\varphi JX, JX).$$

We suppose that $\mathfrak{g}$ has an inner product with the following property: At each point of N the tangent space can be identified with a subspace of $\mathfrak{g}$, the orthogonal of the isotropy subalgebra. This allows us to view J_x as an endomorphism of $\mathfrak{g}$ by extending it as zero on the isotropy subalgebra. We require this extension to be skew symmetric for every x in N. We shall refer to such homogeneous Kähler manifolds as admissible. They include those manifolds with $\mathfrak{g}$ semisimple such as generalized flag manifolds and Hermitian symmetric spaces (the negative of the Killing form will provide such an inner product). We fix such an inner product and denote the sum of the eigenvalues of Q with respect to the inner product by $\operatorname{Tr} Q$.

Lemma 1. *If N is an admissible homogeneous Kähler manifold, then* $\operatorname{Tr} Q = 0$.

Proof. Take any point $x \in N$ and pick an orthonormal basis for $\mathfrak{g}$ of the form $Z_1, \cdots, Z_k, X_1, \cdots, X_n, JX_1, \cdots, JX_n$ with $Z_1, \cdots, Z_k$ a basis for the isotropy subalgebra of x. Thus the latter all vanish at x. Since $J_\varphi X = 0$ for a Killing field and J commutes with ∇ on a Kähler manifold we have

$$
\begin{aligned}
J_\varphi JX &= J\nabla^*\nabla X - \operatorname{Ric}^\varphi(JX) \\
&= JJ_\varphi X + J\operatorname{Ric}^\varphi(X) - \operatorname{Ric}^\varphi(JX) \\
&= J\operatorname{Ric}^\varphi(X) - \operatorname{Ric}^\varphi(JX).
\end{aligned} \tag{1}
$$

Using the above formula for $J_\varphi JX$ we see that summing over the other terms in the basis we get pair-wise cancellation showing that in fact the integrand of $\operatorname{Tr} Q$ vanishes identically.

Lemma 2. *If φ is a stable harmonic map of a Riemannian manifold into an admissible homogeneous Kähler manifold N, then*

$$[\operatorname{Ric}^\varphi, J] = 0. \tag{2}$$

Proof. If φ is stable then Q is nonnegative, but its trace is zero by Lemma 1 and hence Q is zero. Thus $H_\varphi(JX, JX) = 0$ for all Killing fields. On the other hand $H_\varphi(Y, Y) \geq 0$ for all vector fields and so we have the Cauchy-Schwartz inequality

$$H_\varphi(JX, Y)^2 \leq H_\varphi(JX, JX) H_\varphi(Y, Y) = 0.$$

Thus $H_\varphi(JX, Y) = 0$ for all vector fields and hence $J_\varphi JX = 0$. Then (1) and the fact that the Killing fields span the tangent space at each point imply the lemma.

It is possible to average over Killing fields in the domain instead of the range. This gives a condition satisfied by stable harmonic maps from a Hermitian symmetric space to any Riemannian manifold. In [11] Ohnita derives such a condition by using a standard minimal immersion. In fact the two points of view are equivalent and yield Lemma 3 below. To state the lemma we assume M is Hermitian symmetric and for $x \in M$ consider the map $A_x : S^2\mathfrak{p} \to \mathfrak{k}$ ($\mathfrak{k}$ the Lie algebra of the stabilizer K of x, $\mathfrak{p}$ its orthogonal complement) defined by

$$A(X, Y) = [X, J^M Y].$$

A is clearly K-equivariant and surjective since $[\mathfrak{p},\mathfrak{p}] = \mathfrak{k}$, because $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is simple. Hence the adjoint A^* is injective and gives rise to a copy of the adjoint representation of $\mathfrak{k}$ in $S^2\mathfrak{p}$. The components of A relative to a basis of $\mathfrak{k}$ are symmetric bilinear forms on $\mathfrak{p}$. Likewise the components of $\nabla d\varphi_x$ relative to a basis of $T_{\varphi(x)}N$ are also symmetric bilinear forms. The condition on a stable harmonic map is the following:

Lemma 3 (*Ohnita*). *Let $\varphi: M \to N$ be a stable harmonic map of a compact simple Hermitian symmetric space into any Riemannian manifold. Then at each point of M the components of A and the components of $\nabla d\varphi$ are mutually orthogonal.*

Proof. We write φ as the composition $\varphi = \varphi \cdot \mathrm{id}$ and use the formula of Ohnita and Pluzhnikov [13] for the Jacobi operator of a composition:

$$J_\varphi(d\varphi(X)) = d\varphi(J_{\mathrm{id}}X) - 2\sum \nabla d\varphi(X_i, \nabla_{X_i}X).$$

If X is a Killing field then $J_{\mathrm{id}}J^M X = 0$ as in the proof of Lemma 2 since the identity map of a Kähler manifold is holomorphic, hence stable. If we now define a quadratic form on the Lie algebra of Killing fields of M by

$$\begin{aligned} Q'(X,X) &= H_\varphi(J^M X, J^M X) \\ &= -2\int_M \sum(\nabla d\varphi(X_i, J^M\nabla_{X_i}X), d\varphi(J^M X))\ , d\,\mathrm{vol}, \end{aligned} \tag{3}$$

and if M is hermitian symmetric, then given any point we can pick a basis for the Killing fields such that each basis element or its covariant derivative vanishes there. Thus (3) shows Q' has trace zero. Now the argument proceeds as in Lemma 2 to deduce that when φ is stable we have

$$\sum \nabla d\varphi(X_i, J^M\nabla_{X_i}X) = 0$$

for all Killing fields X. If we identify $\mathfrak{p}$ with T_xM then the covariant derivative can be written as the bracket $[X_i, (1-P)X]$, where P is the projection onto $\mathfrak{p}$. A short calculation now shows

$$\sum \nabla d\varphi(X_i, X_j)(A(X_i, X_j), X) = 0$$

and hence the result.

An application of this result will be made in §4. For now we turn to the examination of (2) in the case where the range is Hermitian symmetric. Fix $x \in N$ and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding symmetric or Cartan decomposition which satisfies

$$[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}.$$

Then $\mathfrak{p}$ may be identified with the tangent space at x, and J_x extends as zero on $\mathfrak{k}$ to give a derivation of $\mathfrak{g}$. From this point of view the curvature of N is given by

$$R^N(X,Y) = -\mathrm{ad}[X,Y], \qquad X, Y \in \mathfrak{p}.$$

Suppose that $x = \varphi(y)$, and let $e_1, \cdots, e_m$ be an orthonormal basis for T_yM. Put

$$\varphi_i = d\varphi(e_i), \qquad \varphi_i \in \mathfrak{p};$$

then

$$\mathrm{Ric}^\varphi = -\sum \mathrm{ad}\,\varphi_i \cdot \mathrm{ad}\,\varphi_i.$$

Since J is a derivation, it has the form $\mathrm{ad}\,j$ for some element j of $\mathfrak{k}$ and so

$$\begin{aligned} [\mathrm{Ric}^\varphi, J] &= -\sum [\mathrm{ad}\,\varphi_i \cdot \mathrm{ad}\,\varphi_i, \mathrm{ad}_j] \\ &= -\sum \mathrm{ad}\,\varphi_i \cdot [\mathrm{ad}\,\varphi_i, \mathrm{ad}\,j] + [\mathrm{ad}\,\varphi_i, \mathrm{ad}\,j] \cdot \mathrm{ad}\,\varphi_i \\ &= -\sum \mathrm{ad}\,\varphi_i \cdot \mathrm{ad}[\varphi_i, j] + \mathrm{ad}[\varphi_i, j] \cdot \mathrm{ad}\,\varphi_i \\ &= \sum \mathrm{ad}\,\varphi_i \cdot \mathrm{ad}(J\varphi_i) + \mathrm{ad}(J\varphi_i) \cdot \mathrm{ad}\,\varphi_i. \end{aligned} \tag{4}$$

On the other hand, consider $[d\varphi \cdot d\varphi^t, J]$:

$$\begin{aligned} [d\varphi \cdot d\varphi^t, J]X &= \sum \{\langle d\varphi^t JX, e_i\rangle \varphi_i - \langle d\varphi^t X, e_i\rangle J\varphi_i\} \\ &= -\sum \{\langle X, J\varphi_i\rangle \varphi_i + \langle X, \varphi_i\rangle J\varphi_i\}. \end{aligned}$$

If we identify $\mathrm{End}(\mathfrak{p})$ with $\mathfrak{p} \otimes \mathfrak{p} \subset \mathfrak{g} \otimes \mathfrak{g}$ via the Killing form, then

$$[d\varphi \cdot d\varphi^t, J] = -\sum J\varphi_i \otimes \varphi_i + \varphi_i \otimes J\varphi_i. \tag{5}$$

Finally, we define a linear operator $T: \mathfrak{g} \times \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ by

$$T(\alpha \otimes \beta) = \mathrm{ad}\,\alpha \cdot \mathrm{ad}\,\beta,$$

and conclude that

$$T([d\varphi \cdot d\varphi^t, J]) = -[\mathrm{Ric}^\varphi, J].$$

Proposition 1. *If $\mathfrak{g}$ is simple, then T has kernel contained in $\bigwedge^2 \mathfrak{g}$.*

Proof. See the appendix.

Corollary 1. *If φ is a stable harmonic map into an irreducible Hermitian symmetric space of compact type, then $[d\varphi \cdot d\varphi^t, J] = 0$.*

Proof. Since φ is stable, $[\mathrm{Ric}^\varphi, J] = 0$, so $T[d\varphi \cdot d\varphi^t, J] = 0$. But (5) shows that $[d\varphi \cdot d\varphi^t, J]$ lies in the symmetric part of $\mathfrak{g} \otimes \mathfrak{g}$, hence Proposition 1 gives the result.

3. Applications

We now examine the condition on φ given by the lemma. Since $d\varphi \cdot d\varphi^t$ and $d\varphi$ have the same image, it follows that the image of $d\varphi$ is J-stable and hence of even dimension.

Suppose $d\varphi$ is somewhere injective, and let $U \subset M$ be the open set of all such points. Then φ is locally a diffeomorphism of U with a submanifold of N which is necessarily a complex submanifold. Hence J transports to U to give it an integrable complex structure $\tilde{J} = d\varphi^{-1} J d\varphi$ with respect to which φ is holomorphic. Further

$$\begin{aligned}(\tilde{J} d\varphi^t Y, X) &= (d\varphi^{-1} J d\varphi \, d\varphi^t Y, X) = (d\varphi^t J Y, X) \\ &= -(Y, J d\varphi X) = -(d\varphi^t Y, \tilde{J} X).\end{aligned}$$

Since $d\varphi$ is injective, $d\varphi^t$ is surjective and hence $\tilde{J}$ is Hermitian. This shows:

Corollary 2. *Under the assumptions of Corollary* 1 *with $d\varphi$ somewhere injective, $(U, \tilde{J})$ is a Hermitian manifold with respect to the metric on M, and φ is a holomorphic map:*

$$d\varphi \cdot \tilde{J} = J \cdot d\varphi. \tag{6}$$

Theorem 1. *Let $\varphi: M \to N$ be a stable harmonic map of a closed Riemann surface into a compact simple Hermitian symmetric space. Then φ is $\pm$holomorphic.*

Proof. Either $d\varphi$ has everywhere rank zero (and so φ is constant and we are done) or else $d\varphi$ has rank two on a nonempty open set U. Then Corollary 2 implies φ is holomorphic with respect to $\tilde{J}$. Since both $\tilde{J}$ and J^M are Hermitian and the domain is 2-dimensional, $\tilde{J} = \pm J$ at each point of U and hence at least one of these signs holds on a nonempty open set. Then φ is holomorphic or antiholomorphic on an open set and everywhere by Siu's Unique Continuation Theorem [18]. q.e.d.

We continue with our deductions from Proposition 1: Covariantly differentiating (6) we have

$$\nabla d\varphi \cdot \tilde{J} + d\varphi \cdot \nabla \tilde{J} = \nabla J \cdot d\varphi + J \cdot \nabla d\varphi.$$

Since N is Kähler, $\nabla J = 0$. Taking the trace,

$$\sum \nabla d\varphi(e_i, \tilde{J} e_i) + d\varphi((\nabla_{e_i} \tilde{J}) e_i) = \sum J \cdot \nabla d\varphi(e_i, e_i) = 0,$$

since φ is harmonic. Using the symmetry of $\nabla d\varphi$ and the fact that we can take an orthonormal basis of the form $e_1, \tilde{J} e_1, \cdots$, we see that the first summation on the left-hand side is also zero. Since φ is an immersion, we conclude

$$\operatorname{Trace} \nabla \tilde{J} = 0.$$

This can be reformulated in terms of the Kähler form as:

Corollary 3. *The Hermitian structure* $(U, \tilde{J})$ *has coclosed Kähler form.*

Theorem 2. *If* $\varphi: M \to N$ *is a stable harmonic immersion of a Riemannian manifold into a compact simple Hermitian symmetric space, then* M *is even dimensional, admits a complex structure which is Hermitian for the given metric, and has coclosed Kähler form.* φ *is holomorphic for this complex structure, and* φ *is an absolute energy minimizer in its homotopy class. If* M *has a given Hermitian structure with coclosed Kähler form, and* φ *is homotopic to a* $\pm$*holomorphic map for this given structure, then* φ *is itself* $\pm$*holomorphic.*

Proof. The open set U above is now all of M, so Corollaries 2 and 3 give M the Hermitian structure with coclosed Kähler form with respect to which φ is holomorphic. The result of Lichnerowicz cited above now implies φ is an absolute energy minimizer in its homotopy class. The last part of the theorem is standard. q.e.d.

If U has dimension 4, then a Kähler form is anti-self-dual with respect to the orientation defined by the complex structure and coclosed if and only if closed. Hence

Corollary 4. *If* $\dim M = 4$, *then* $(U, \tilde{J})$ *is a Kähler manifold.*

In general we know little about the set U. Suppose however that the Riemannian structure on M is real analytic and U is not empty. Then φ is real analytic and so U is a dense open set whose complement is a real analytic subvariety. We cannot assert that U is connected, nevertheless we claim that in this situation $\tilde{J}$ extends to all of M.

Theorem 3. *Let* $\varphi: M \to N$ *be a stable harmonic map from a real analytic four-dimensional Riemannian manifold* M *to a compact simple Hermitian symmetric space* N, *and suppose that there is a point where* $d\varphi$ *has rank at least three. Then* M *has a unique Kähler structure* $\tilde{J}$ *with respect to which* φ *is holomorphic.*

Proof. By our previous remarks we have a nonempty open set U which is dense in M on which $\tilde{J}$ is defined by the condition that φ be holomorphic, and $\tilde{J}$ is Kähler, by Corollary 4. We have only to show that $\tilde{J}$ extends to M. The fact that U is dense then guarantees the other properties extend from U to M. Take a point x in $M \backslash U$ and a geodesically convex disc D around x. $D \cap U$ is nonempty, so pick a point y in it and consider $\tilde{J}_y$. Construct an almost complex structure J' on D by parallel transport of $\tilde{J}_y$ along the unique geodesic joining y to each point of D. J' is real analytic and agrees with $\tilde{J}$ at least on a convex neighborhood V of y in $U \cap D$ as a result of $\tilde{J}$ being parallel. Since φ is holomorphic with respect to $\tilde{J}$ on U, it is holomorphic with respect to J' on V. But φ and J' are both real analytic, hence φ is holomorphic with respect to J' on all of D. Thus we have both

$$d\varphi \cdot \tilde{J} = J^N \cdot d\varphi, \qquad d\varphi \cdot J' = J^N \cdot d\varphi$$

on $D \cap U$. Since $d\varphi$ is injective on U, it follows that J' and $\tilde{J}$ agree on $D \cap U$. It is now easy to see that these extensions of $\tilde{J}$ all fit together to give a globally defined $\tilde{J}$ on all of M.

Corollary 5. *Let $\varphi: M \to N$ be a stable harmonic map from a real analytic Riemannian 4-manifold to a compact simple Hermitian symmetric space. Then M not Kählerian implies that $d\varphi$ has rank everywhere less than or equal to 2.*

There are topological restrictions on a compact 4-manifold for it to admit a Kähler structure (it must be orientable, have even first betti number and nonzero second betti number) so this corollary is quite restrictive.

In [2] Burns and de Bartolomeis obtained results on the stability of harmonic maps of Kähler surfaces into CP^n by employing the twistor space on the domain. By assuming that the rank is somewhere maximal we can obtain similar results for arbitrary compact simple Hermitian symmetric range. The result follows from:

Proposition 2. *Let (M, g) be a Riemannian 4-manifold with two Kähler structures J' and J''. Then either J' and J'' commute or they generate a 2-sphere of Kähler structures so that M is Ricci-flat. In the case where they commute there are two possibilities: If they define the same orientation then $J' = \pm J''$, while if they are oppositely oriented then M is covered by a product of Riemann surfaces with J' given by the product complex structure and J'' by reversing one of the factors.*

Proof. We use the following well-known and easily verified facts. J' and J'' are elements of so(4). If they are oppositely oriented, they commute and belong to different simple factors. If they belong to different simple factors, then they are oppositely oriented and commute.

If J' and J'' have the same orientation and commute, they necessarily belong to the same simple factor which has rank one. Hence they are proportional so $J' = \pm J''$. If they have the opposite orientation and commute, then $F = J' \cdot J''$ has two-dimensional eigendistributions which are preserved by the Levi-Civita connection. Hence both eigendistributions define parallel foliations by complex curves, and M is covered by a product of the form described in the proposition. This leaves us with the case where J' and J'' do not commute and so have the same orientation. It follows that at any point they belong to the same simple factor of so(4) and so generate an su(2) subalgebra. This is obviously stable under covariant differentiation, so we are in the situation of SU(2) holonomy.

Corollary 6. *Let M be a compact connected Kähler surface with real analytic metric, and let $\varphi: M \to N$ be a stable harmonic map into a compact simple Hermitian symmetric space with rank somewhere greater than two. Then either φ is $\pm$holomorphic or M is covered by a product of Riemann surfaces and the lift of φ is holomorphic on one factor and antiholomorphic on the other. The only remaining possibility is that M is Ricci flat and φ is holomorphic with respect to another of the 2-sphere of complex structures carried by M.*

Proof. By Theorem 3 there is a Kähler structure $\tilde{J}$ on M with respect to which φ is holomorphic. The analysis in Proposition 2 now gives the result.

4. Maps from symmetric spaces

In [11] Ohnita studies a class of maps from Kähler manifolds into Riemannian manifolds which he calls pluriharmonic, namely, they are maps φ for which the $(1,1)$ part of the second fundamental form $\beta = \nabla d\varphi$ is zero. In [15] these were called $(1,1)$-geodesic maps by analogy with $\beta = 0$ for totally geodesic maps. It is shown in [15] that a map is $(1,1)$-geodesic if and only if it is harmonic whenever it is restricted to any germ of a Riemann surface in the domain. Ohnita has proven

Theorem 4. *Let $\varphi: CP^n \to N$ be a stable harmonic map where N is any Riemannian manifold. Then φ is $(1,1)$-geodesic.*

Proof. Since CP^n is a compact simple Hermitian symmetric space, Lemma 3 implies that β has components which are orthogonal to the image of A^*. But it is easy to see that in this case the image of A^* is the set of bilinear forms of type $(1,1)$ (they coincide precisely with the adjoint representation of K) and hence the result.

Lemma 4. *Let $\varphi: M \to N$ be a $(1,1)$-geodesic map from a generalized flag manifold of a compact simple Lie group equipped with any invariant Kähler metric into any Riemannian manifold. Then $\varphi^* h^{(2,0)} = 0$.*

Proof. Since the complex structure is invariant, the $(1,0)$ parts of Killing vector fields are holomorphic. Thus let X, Y be Killing fields and let X', Y' denote their $(1,0)$ parts. Then

$$\varphi^* h^{(2,0)}(X, Y) = \varphi^* h(X', Y').$$

But if Z is any $(0,1)$ vector, we have

$$Z\varphi^* h(X', Y') = h(\nabla_Z d\varphi(X'), d\varphi(Y')) + h(d\varphi(X'), \nabla_Z d\varphi(Y')).$$

However φ $(1,1)$-geodesic implies

$$\nabla_Z d\varphi(X') = d\varphi(\nabla_Z X'),$$

which is zero since X' is holomorphic. Hence $\varphi^*h(X', Y')$ is a holomorphic function, and therefore a constant. But the Killing fields on a flag manifold are Hamiltonian when the manifold is considered as a homogeneous symplectic manifold with respect to its Kähler form. Since the flag manifold is compact, any Hamiltonian function necessarily has critical points, and all Killing fields have zeros. Thus $\varphi^*h(X', Y') = 0$. This proves the lemma since the Killing fields span the tangent spaces at every point.

Remark. Ohnita's proof uses a Bochner-type vanishing theorem and is valid on any Kähler-Einstein manifold with positive scalar curvature. On a flag manifold we have a cone of invariant Kähler metrics with Einstein metrics on the axis, so the above gives an alternative proof of Ohnita's result in the case of a flag manifold.

Lemma 4 implies that φ^*h is a $(1,1)$ form and so may be viewed as an exterior 2-form ω on M given by $\omega(X, Y) = \varphi^*h(X, J^M Y)$.

Lemma 5 (*Ohnita*). *Under the same conditions as in Lemma* 4 *the* 2-*form* ω *is closed.*

Proof. Since ω is of type $(1,1)$ and real we only have to show that the part of $d\omega$ of type $(2,1)$ is zero. Thus if X, Y are $(1,0)$ vectors and Z is $(0,1)$, then we have

$$\begin{aligned}
\mathrm{i}d\,\omega(X,Y,Z) &= X\varphi^*h(Y,Z) - Y\varphi^*h(X,Z) - \varphi^*h([X,Y],Z) \\
&\quad - \varphi^*h([X,Z],Y) + \varphi^*h([Y,Z],X) \\
&= \varphi^*h(\nabla_X Y,Z) + \varphi^*h(Y,\nabla_X Z) - \varphi^*h(\nabla_Y X,Z) \\
&\quad - \varphi^*h(X,\nabla_Y Z) - \varphi^*h([X,Y],Z) - \varphi^*h(\nabla_X Z,Y) \\
&\quad + \varphi^*h(\nabla_Y Z,X) \\
&= 0. \quad \text{q.e.d.}
\end{aligned}$$

Suppose the 2-form ω is nowhere of maximum rank. If $m = \dim M$, then we have $\omega^m = 0$ and hence the cohomology class $[\omega]^m$ also vanishes. But for a compact simple Hermitian symmetric space the cohomology in degree two is generated by a single form whose mth power is nonzero, hence $[\omega] = 0$. Since ω is a nonnegative $(1,1)$-form, $\omega = 0$ and φ is constant. Thus we have proved

Lemma 6. *A* $(1,1)$-*geodesic map of a compact simple Hermitian symmetric space is an immersion on a nonempty open set.*

Combining Theorem 4, Lemma 6 and a result of Udagawa [20], Ohnita [11] then shows that when $N = CP^n$, φ must be $\pm$holomorphic. We shall prove that this result remains true when the target is any compact simple Hermitian symmetric space.

From Theorem 4 and Lemma 6 we have an open set U on which φ is an immersion, and if φ is stable as a harmonic map then we have the Hermitian structure $\tilde{J}$ on U from Corollary 2 when N is a compact simple Hermitian symmetric space. Hence we have

Proposition 3. *If $\varphi: CP^n \to N$ is a nonconstant stable harmonic map into a compact simple Hermitian symmetric space, then φ is $(1,1)$-geodesic and is an immersion on a nonempty open set U, and $\tilde{J}$ is a Hermitian structure on U with respect to which φ is holomorphic.*

Now consider any Hermitian symmetric space M and a second Hermitian structure $\tilde{J}$ defined on an open set U, but compatible with the standard metric on M. This is the case in Proposition 2 with $M = CP^n$. We can view $\tilde{J}$ as a section over U of the twistor space $Z = J(M, g)$ of complex structures on the tangent spaces of M compatible with the metric. Z itself carries a natural almost complex structure, and the fact that $\tilde{J}$ is integrable forces its image to lie in the zero set of the Nijenhuis tensor on Z. This set has been completely determined by Burstall and Rawnsley who use it to show [4, Theorem 5.6] that any Hermitian structure on a simple Hermitian symmetric space necessarily commutes with the invariant Hermitian structure:

Proposition 4. *Let M be a compact simple Hermitian symmetric space, and let J^M denote its standard invariant complex structure. If $\tilde{J}$ is a second Hermitian structure defined on an open set in M, then $\tilde{J}$ and J^M commute.*

Theorem 5. *Let $\varphi: CP^n \to N$ be a stable harmonic map into a compact simple Hermitian symmetric space. Then φ is $\pm$-holomorphic.*

Proof. Let $M = CP^n$. By Proposition 3 it is enough to show that $\tilde{J} = \pm J^M$. We claim that $\tilde{J}$ is in fact parallel on U. To see this we observe that since $\tilde{J}$ and J^M commute, we can diagonalize them simultaneously. That is, if T' denotes the $(1,0)$ vectors for J^M and T^+ the $(1,0)$ vectors for $\tilde{J}$, then there are orthogonal subbundles V and W of TU^c such that

$$T' = V + W, \qquad T^+ = V + \overline{W}.$$

Since $\tilde{J}$ is integrable we have $\nabla_Z T^+ \subset T^+ \ \forall Z \in T^+$, and since J^M is parallel we thus have

$$\nabla_Z V \subset V, \quad \nabla_Z \overline{W} \subset \overline{W} \quad \forall Z \in T^+. \tag{7}$$

Differentiating $d\varphi \cdot \tilde{J} = J^N \cdot d\varphi$ gives

$$J^N \cdot \beta = \beta \cdot \tilde{J} + d\varphi(\nabla \tilde{J}).$$

Since φ is $(1,1)$-geodesic by Theorem 4, the first two terms in the above equation have vanishing $(1,1)$ part, and since φ is an immersion on U we obtain

$$\nabla_{\overline{V}}\tilde{J}V = \nabla_W \tilde{J}\overline{W} = 0.$$

Hence $\nabla_{\overline{V}}V \subset T^+$. But J^M being parallel implies the left-hand side is in T' so $\nabla_{\overline{V}}V \subset V$.

A similar argument shows $\nabla_W \overline{W} \subset \overline{W}$, and combining these with (7) we deduce that both V and W are parallel. Thus $\tilde{J}$ is a Kähler complex structure on U with respect to the Fubini-Study metric on CP^n. Since CP^n is isotropy irreducible and J^M spans the center of the holonomy Lie algebra at each point, it follows that $\tilde{J} = \pm J^M$ at some point and, by parallel transport, a constant sign holds on a nonempty open set. Thus φ is $\pm$holomorphic on a nonempty open set and hence on all of CP^n by Siu's Unique Continuation Theorem [18].

Alternatively, $\tilde{J}$ is a horizontal section of a twistor bundle $G_r(T')$ over CP^n. But when $0 < r < n$, the horizontal distribution has an integrability tensor [14] which is sufficiently nondegenerate that any horizontal map has rank at most two. This cannot happen for a section and hence r is 0 or n. The two extreme values correspond to V or W are zero and hence $\tilde{J} = \pm J^M$.

Appendix

Proof of Proposition 1. Consider the map $T\colon \mathfrak{g} \otimes \mathfrak{g} \to \operatorname{End} \mathfrak{g}$ given by

$$T(A \otimes B) = \operatorname{ad} A \cdot \operatorname{ad} B.$$

Our aim is to show T is injective on the symmetric part $S^2\mathfrak{g}$ of $\mathfrak{g} \otimes \mathfrak{g}$. Fix an invariant inner product $(\ , \ )$ on $\mathfrak{g}$, and let X_i be an orthonormal basis. Then

$$\begin{aligned}(T(A \otimes B)X, Y) &= -([B, X], [A, Y]) \\ &= -\sum([B, X], X_i)(X_i, [A, Y]) \\ &= -\sum([X_i, B], X)([X_i, A], Y) \\ &= -\sum(\operatorname{ad} X_i B \otimes \operatorname{ad} X_i A, X \otimes Y).\end{aligned}$$

But

$$-\sum_i \operatorname{ad} X_i \otimes \operatorname{ad} X_i = \tfrac{1}{2}\sum(\operatorname{ad} X_i \otimes 1 + 1 \otimes \operatorname{ad} X_i)^2$$
$$- (\operatorname{ad} X_i)^2 \otimes 1 - 1 \otimes (\operatorname{ad} X_i)^2$$
$$= \tfrac{1}{2}(C_2 - C_1 \otimes 1 - 1 \otimes C_1),$$

where C_i denotes the Casimir operator of $\mathfrak{g}$ in the ith tensor power of the adjoint representation. Hence

$$(T(A \otimes B)X, Y) = \tfrac{1}{2}(\{C_2 - C_1 \otimes 1 - 1 \otimes C_1\}(B \otimes A), X \otimes Y),$$

from which it follows that T has the same kernel as $C_2 - C_1 \otimes 1 - 1 \otimes C_1$. This allows us to use representation theory to study T.

Since $\mathfrak{g}$ is simple, its adjoint representation is irreducible, so Schur's Lemma [7, p. 26] implies C_1 is a multiple of the identity. We can scale the inner product to make this multiple equal to 1 (this corresponds with taking (,) to be the negative of the Killing form). Hence

$$T = C_2 - 2.$$

The invertibility of T thus reduces to whether or not 2 is an eigenvalue of C_2. This we shall determine in the next lemma using the structure theory of simple Lie algebras (see [7]):

Lemma A. *Let $\mathfrak{g}$ be a simple Lie algebra, and C_2 the Casimir operator on $S^2\mathfrak{g}$ defined by the Killing form. Then 2 is not an eigenvalue.*

Proof. Fix a Cartan subalgebra $\mathfrak{a}$ and a positive root system. Let θ be the highest root, and 2ρ the sum of the positive roots. Since θ is the highest weight of $\mathfrak{g}$, any highest weight of $S^2\mathfrak{g}$ has the form $\theta + \alpha$ for some root α [7, p. 142].

On the other hand, in any irreducible representation of highest weight λ, the Casimir operator C has eigenvalue $(\lambda + 2\rho, \lambda)$ [7, p. 134]. Thus C_1 has eigenvalue $(\theta + 2\rho, \theta) = 1$, and C_2 will have eigenvalue $(\theta + \alpha + 2\rho, \theta + \alpha)$ on the above eigenspace. Let

$$A = (\theta + \alpha + 2\rho, \theta + \alpha) - 2 = 2(\alpha, \theta) + (\alpha + 2\rho, \alpha) - (\theta + 2\rho, \theta).$$

If $\alpha = \theta$, then $A = 2|\theta|^2$ which is certainly nonzero, so we can restrict attention to $\alpha \neq \theta$ and hence $(\alpha + 2\rho, \alpha) < (\theta + 2\rho, \theta)$ [7, p. 71]. Then A can only vanish if $(\alpha, \theta) > 0$, so $\beta = \theta - \alpha$ is also a root, necessarily positive.

Suppose $|\alpha| \neq |\beta|$, say $|\alpha| > |\beta|$ (by symmetry it is enough to consider this case). Then $|\theta|^2 = |\alpha|^2 + |\beta|^2 + 2(\alpha, \beta)$, and θ and α are long, so that $0 = |\beta|^2 + 2(\alpha, \beta)$. Hence $(\beta, 2\alpha/|\alpha|^2) = -|\beta|^2/|\alpha|^2 \notin Z$. This is impossible, thus $|\alpha| = |\beta|$. A short calculation using $\theta = \alpha + \beta$ shows that

$$A = |\beta|^2 - 2(\rho, \beta).$$

So A can only vanish if $(\rho, 2\beta/|\beta|^2) = 1$. Let $\alpha_1, \cdots, \alpha_n$ be the simple roots; then $(\rho, 2\alpha_i/|\alpha_i|^2) = 1$ for all i [7, p. 70]. Hence if $\beta = \sum_i n_i \alpha_i$, then

$$(\rho, \beta) = \sum_i n_i(\rho, \alpha_i) = \tfrac{1}{2} \sum_i n_i |\alpha_i|^2,$$

so

$$|\beta|^2 = \sum_i n_i |\alpha_i|^2.$$

If β is not simple, then it follows from this formula that it must be a long root, and the α_i occurring in the sum must all be short roots. In the case of G_2 we have only one short simple root, and the ratio $|\beta|^2/|\alpha_i|^2 = 3$ would force β to be three times the short simple root. This cannot happen, so the ratio must be 2. The only way this can happen for nonsimple β is if it is the sum of two short simple roots. But the latter case then implies that α is long so that the simple roots are orthogonal. Thus their difference is also a root. But this never happens, so we conclude that β is itself simple.

We have thus shown that the highest weight of the form $\theta + \alpha$ must have $\theta = \beta + \alpha$ with β simple. Suppose the weight $\theta + \alpha$ occurs in $S^2 g$ with multiplicity larger than one, that is, $\theta + \alpha = \delta + \gamma$ for two roots with neither δ nor γ equal to θ. Then both δ and γ can be written as θ minus at least one simple root, and so $\delta + \gamma$ is 2θ minus at least two simple roots. On the other hand $\theta + \alpha = 2\theta - \beta$ which is 2θ minus just one simple root. Thus $\theta + \alpha \neq \delta + \gamma$. It now follows that the only way $\theta + \alpha$ can arise as a highest weight in $\operatorname{Ker} T$ is on the product of $\mathfrak{g}_\theta$ and $\mathfrak{g}_\alpha$.

However $\mathfrak{g}_\theta \vee \mathfrak{g}_\alpha$ is never a highest weight space since

$$e_\beta \cdot (\mathfrak{g}_\theta \vee \mathfrak{g}_\alpha) = \mathfrak{g}_\theta \vee e_\beta \cdot \mathfrak{g}_\alpha = \mathfrak{g}_\theta \vee \mathfrak{g}_\theta$$

is not zero. Hence A is always nonzero, which completes the proof.

References

[1] A. Borel, *On the curvature tensor of the Hermitian symmetric manifolds*, Ann. of Math. (2) **71** (1960) 508–521.

[2] D. Burns & P. de Bartolomeis, *Applications harmoniques stables dans P^n*, Ann. Sci. École Norm. Sup. (4) **21** (1988) 159–187.

[3] F. Burstall, J. Rawnsley & S. Salamon, *Stable harmonic 2-spheres in symmetric spaces*, Bull. Amer. Math. Soc. (N.S.) **16** (1987) 274–278.

[4] F. Burstall & J. Rawnsley, *Twistor theory for Riemannian symmetric spaces*, to appear.

[5] E. Calabi & E. Vesentini, *On compact, locally symmetric Kähler manifolds*, Ann. of Math. (2) **71** (1960) 472–507.
[6] J. Eells & L. Lemaire, *Selected topics in harmonic maps*, C.B.M.S. Regional Conf. Ser. in Math., No. 50, Amer. Math. Soc., Providence, RI, 1983.
[7] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in math., Vol. 9, Springer, New York, 1972.
[8] H. B. Lawson & J. Simons, *On stable currents and their application to global problems in real and complex geometry*, Ann. of Math. (2) **98** (1973) 427–450.
[9] A. Lichnerowicz, *Applications harmoniques et variétés kähleriennes*, Sympos. Math. **3** (1970) 341–402.
[10] J. D. Moore, *Compact Riemannian manifolds with positive curvature operators*, Bull. Amer. Math. Soc. (N.S.) **14** (1986) 279–282.
[11] Y. Ohnita, *Pluriharmonicity of stable harmonic maps*, J. London Math. Soc. **35** (1987) 563–568.
[12] Y. Ohnita & S. Udagawa, *Stable harmonic maps from Riemann surfaces to compact Hermitian symmetric spaces*, Tokyo J. Math. **10** (1987) 385–390.
[13] A. Pluzhnikov, *On the minimum of the Dirichlet functional*, to appear.
[14] J. Rawnsley, *On the rank of horizontal maps*, Math. Proc. Cambridge Philos. Soc. **92** (1982) 485–488.
[15] ——, *f-structures, f-twistor spaces and harmonic maps*, Geometry Seminar "Luigi Bianchi"-1984 (E. Vesentini, ed.), Lecture Notes in Math., Vol. 1164, Springer, Berlin, 1985.
[16] Y.-T. Siu, *Some remarks on the complex-analyticity of harmonic maps*, Southeast Asian Bull. Math. **3** (1979) 240–253.
[17] ——, *A curvature characterization of the hyperquadrics*, Duke Math. J. **47** (1980) 641–654.
[18] ——, *The complex analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds*, Ann. of Math. (2) **112** (1980) 73–111.
[19] Y.-T. Siu & S.-T. Yau, *Compact Kähler manifolds of positive bisectional curvature*, Invent. Math. **59** (1980) 189–204.
[20] S. Udagawa, *Minimal immersions of Kähler manifolds into complex space forms*, Tokyo J. Math. **10** (1987) 227–239.
[21] W. Zakrzewski, *Classical solutions of two-dimensional Grassmannian models*, J. Geom. Phys. **1** (1984) 39–63.
[22] J.-Q. Zhong, *The degree of strong nondegeneracy of the bisectional curvature of exceptional bounded symmetric domains*, Proc. 1981 Hangzhou Conf., Birkhäuser, 1984, 127–139.

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